ACT 3230 Actuarial Models II

Exam 3 – Chapter 9

April 8, 2009
4:00 p.m. – 5:15 p.m.

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#8 ii) $Var(\ddot{a}) = 8.253$

1. Given:
   i) $T(x)$ and $T(y)$ are independent
   ii)

k	0	1	2
q_{x+k}	0.08	0.09	0.10
q_{y+k}	0.10	0.15	0.20

Calculate $_{2|}q_{\overline{xy}}$.

<u>Solution</u>:

$_2p_x = (p_x)(p_{x+1}) = (1-0.08)(1-0.09) = 0.8372$
$_2p_y = (p_y)(p_{y+1}) = (1-0.10)(1-0.15) = 0.765$
$_3p_x = (p_x)(p_{x+1})(p_{x+2}) = (1-0.08)(1-0.09)(1-0.10) = 0.75348$
$_3p_y = (p_y)(p_{y+1})(p_{y+2}) = (1-0.10)(1-0.15)(1-0.20) = 0.612$

$$
\begin{aligned}
{}_{2|}q_{\overline{xy}} &= {}_3q_{\overline{xy}} - {}_2q_{\overline{xy}} \\
&= {}_3q_x \cdot {}_3q_y - {}_2q_x \cdot {}_2q_y \\
&= (1-0.75348)(1-0.612) - (1-0.8372)(1-0.765) \\
&= 0.09564976 - 0.038258 \\
&= \mathbf{0.05739176}
\end{aligned}
$$

Alternative 1:

$$
\begin{aligned}
{}_{2|}q_{\overline{xy}} &= {}_2p_{\overline{xy}} - {}_3p_{\overline{xy}} \\
&= ({}_2p_x + {}_2p_y - {}_2p_x\,{}_2p_y) - ({}_3p_x + {}_3p_y - {}_3p_x\,{}_3p_y) \\
&= [0.8372 + 0.765 - (0.8372)(0.765)] - [0.75348 - 0.612 - (0.75348)(0.612)] \\
&= 0.961742 - 0.90435024 \\
&= \mathbf{0.05739176}
\end{aligned}
$$

Alternative 2:

$$
\begin{aligned}
{}_{2|}q_{\overline{xy}} &= {}_{2|}q_x + {}_{2|}q_y - {}_{2|}q_{xy} \\
&= {}_2p_x\, q_{x+2} + {}_2p_y\, q_{y+2} - {}_2p_x\, {}_2p_y\,(1 - p_{x+2}\, p_{y+2}) \\
&= (0.8372)(0.1) + (0.765)(0.20) - (0.8372)(0.765)[1-(0.9)(0.8)] \\
&= 0.08372 + 0.153 - 0.17932824 \\
&= \mathbf{0.05739176}
\end{aligned}
$$

Alternative 3:

$$
{}_{2|}q_{\overline{xy}} = {}_2q_x\, {}_2p_y\, q_{y+2} + {}_2q_y\, {}_2p_x\, q_{x+2} + {}_2p_x\, {}_2p_y\, q_{x+2}\, q_{y+2}
$$

$= (1 - 0.8372)(0.765)(0.20) + (1 - 0.765)(0.8372)(0.10) + (0.8372)(0.765)(0.10)(0.20)$
$= 0.0249084 + 0.0196742 + 0.01280916$
$= \mathbf{0.05739176}$

But note it does <u>not</u> equal:

$$_{2|}q_{\overline{xy}} \neq {}_2p_{\overline{xy}}\; q_{\overline{x+2:y+2}}$$
$$= ({}_2p_x + {}_2p_y - {}_2p_x\, {}_2p_y)\,(q_{x+2}\, q_{y+2})$$
$$= (0.961742)(0.10)(0.20)$$
$$= 0.01923484$$

2. Jack and Jill are independent lives aged 90 and 95, respectively. Both have mortality that follows DeMoivre's Law with $\omega = 100$.

   Calculate the probability that Jack's curtate age at death (i.e. the largest full integer age reached) is greater than Jill's.

   <u>Solution</u>:

   If Jill dies at age 95, then Jack must survive to at least 96. If Jill dies at age 96, then Jack must survive to at least 97, etc.

   The overall probability is:

$$q_{95}\cdot {}_6p_{90} + {}_{1|}q_{95}\cdot {}_7p_{90} + {}_{2|}q_{95}\cdot {}_8p_{90} + {}_{3|}q_{95}\cdot {}_9p_{90}$$
$$= \frac{1}{5}\cdot\frac{4}{10} + \frac{1}{5}\cdot\frac{3}{10} + \frac{1}{5}\cdot\frac{2}{10} + \frac{1}{5}\cdot\frac{1}{10} = \frac{1}{5}$$

3. Jack and Jill are independent lives aged 90 and 95, respectively. Both have mortality that follows DeMoivre's Law with $\omega = 100$.

Calculate $\mathring{e}_{\overline{90:95}}$.

Solution:

$\mathring{e}_{\overline{90:95}} = \mathring{e}_{90} + \mathring{e}_{95} - \mathring{e}_{90:95}$

Under DML with $\omega = 100$, $\mathring{e}_{90} = \dfrac{100-90}{2} = 5$ and $\mathring{e}_{95} = \dfrac{100-95}{2} = 2.5$.

$\mathring{e}_{90:95} = \int_0^\infty {}_tp_{90:95}\,dt = \int_0^5 ({}_tp_{90})({}_tp_{95})\,dt$

Note that for the joint-life status expectation, the upper limit of the integral is the *smaller* of the maximum remaining lifetimes of the two individuals. The probability of the joint-life status surviving past that point is 0 (but if you don't recognize this and choose to integrate with the wrong bounds, you will end up with a wrong answer).

$$\mathring{e}_{90:95} = \int_0^5 \left(\frac{10-t}{10}\right)\left(\frac{5-t}{5}\right)dt = \frac{1}{50}\int_0^5 50 - 15t + t^2\,dt = \frac{1}{50}\left[50t - 15\frac{t^2}{2} + \frac{t^3}{3}\right]_0^5$$

$$= \frac{1}{50}\left[50(5) - 15\frac{(5)^2}{2} + \frac{(5)^3}{3}\right] = 104.166666 / 50 = 2.083333333$$

$\mathring{e}_{\overline{90:95}} = 5 + 2.5 - 2.083333333 = \mathbf{5.41667}$

4. For a fully continuous last-survivor whole life insurance of 1 issued to (x) and (y), you are given:
   i) $T(x)$ and $T(y)$ are independent
   ii) $\mu_x(t) = \mu_y(t) = \mu \qquad t > 0$
   iii) $\delta = 0.04$
   iv) Premiums of 0.072 per year, set using the equivalence principle, are payable until the first death.

   Calculate $\mu_x(t)$.

Solution:

At time 0, APV of future premium = APV of future benefits:

$$P\,\bar{a}_{xy} = \bar{A}_{\overline{xy}}$$

$$P\left(\frac{1-\bar{A}_{xy}}{\delta}\right) = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

Re: Under CF assumption, $\bar{A}_x = \dfrac{\mu}{\mu+\delta}$ and $\bar{A}_{xy} = \dfrac{2\mu}{2\mu+\delta}$.

Proof of $\bar{A}_{xy} = \dfrac{2\mu}{2\mu+\delta}$:

$\bar{A}_{xy} = \int_0^\infty v^t\,{}_tp_{xy}\mu_{xy}(t)dt$ Re: With independence, ${}_tp_{xy} = {}_tp_x \cdot {}_tp_y$ and $\mu_{xy}(t) = \mu_x(t) + \mu_y(t)$

$$= \int_0^\infty e^{-\delta t}e^{-\mu t}e^{-\mu t}(\mu+\mu)dt = \int_0^\infty 2\mu \cdot e^{-(2\mu+\delta)t}dt$$

$$= \left.\frac{2\mu \cdot e^{-(2\mu+\delta)t}}{-(2\mu+\delta)}\right|_0^\infty = \frac{2\mu}{2\mu+\delta} \qquad \text{// QED}$$

$$P\left(\frac{1-\bar{A}_{xy}}{\delta}\right) = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

$$P - P\bar{A}_{xy} = \delta\bar{A}_x + \delta\bar{A}_y - \delta\bar{A}_{xy}$$

$$P = 2\delta\bar{A}_x + (P-\delta)\bar{A}_{xy} = 2\delta\left(\frac{\mu}{\mu+\delta}\right) + (P-\delta)\left(\frac{2\mu}{2\mu+\delta}\right)$$

$$P(2\mu+\delta)(\mu+\delta) = 2\mu\delta(2\mu+\delta) + (P-\delta)(2\mu)(\mu+\delta)$$

$$P(2\mu^2 + 3\mu\delta + \delta^2) = 4\mu^2\delta + 2\mu\delta^2 + (0.072 - 0.04)(2\mu^2 + 2\mu\delta)$$

$$.144\mu^2 + .00864\mu + .0001152 = .16\mu^2 + .0032\mu + .064\mu^2 + .00256\mu$$

$$0 = 0.08\mu^2 - 0.00288\mu - 0.0001152$$

$$\mu = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{.00288 \pm \sqrt{(-.00288)^2 - 4(.08)(-.0001152)}}{2(.08)} = \frac{.00288 \pm .00672}{.16}$$

$= 0.06$ or -0.024

Therefore, $\mu =$ **0.06**

5. You are given:
   i) $E[T(x)] = E[T(y)] = 4.0$
   ii) $Cov[T(x), T(y)] = 0.01$
   iii) $Cov[T(xy), T(\overline{xy})] = 0.10$

   Calculate $E[T(\overline{xy})]$.

Solution:

$Cov[T(xy), T(\overline{xy})] = Cov[T(x), T(y)] + (\mathring{e}_x - \mathring{e}_{xy})(\mathring{e}_y - \mathring{e}_{xy})$

$0.10 = 0.01 + (4 - \mathring{e}_{xy})(4 - \mathring{e}_{xy})$

$0.09 = (4 - \mathring{e}_{xy})^2$

$\pm 0.3 = 4 - \mathring{e}_{xy}$

Thus, $\mathring{e}_{xy} = 3.7$ or 4.3.

Since $T(xy) = \min[T(x), T(y)]$, we know $\mathring{e}_{xy} \leq \mathring{e}_x$. So, $\mathring{e}_{xy} = 3.7$.

$\mathring{e}_{xy} + \mathring{e}_{\overline{xy}} = \mathring{e}_x + \mathring{e}_y$

$\mathring{e}_{\overline{xy}} = 4 + 4 - 3.7 = \mathbf{4.3}$

6. For a special fully continuous last-survivor whole life insurance of 1 on (x) and (y), you are given:
   i) The premium is payable until the first death.
   ii) The independent random variables $T^*(x)$, $T^*(y)$, and Z are the components of a common shock model.
   iii) $T^*(x)$ has an exponential distribution with mean 25.
   iv) $T^*(y)$ has an exponential distribution with mean 16.66667.
   v) Z, the common shock random variable, has an exponential distribution with mean 50.
   vi) $\delta = 0.04$

   Calculate the annual benefit premium.

Solution:

Re: for exponential distribution, mean = $1/\mu$
So, $\mu_x = 1/25 = 0.04$, $\mu_y = 1/16.66667 = 0.06$, and $\lambda = 1/50 = 0.02$

APV of prem:

$$P\bar{a}_{xy} = P\int_0^\infty e^{-\delta \cdot t}\,{}_tp_{xy}\,dt = P\int_0^\infty e^{-\delta \cdot t}e^{-\mu_x t}e^{-\mu_y t}e^{-\lambda t}\,dt = P\int_0^\infty e^{-.16t}\,dt = P\left(\frac{1}{.16}\right) = 6.25P$$

Or, just know that under constant force w/ common shock, $\bar{a}_{xy} = \dfrac{1}{\mu_x + \mu_y + \lambda + \delta}$

APV of benefit:

$$\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$$

Under common shock, the shock parameter adds to the force of failure, even in the single lives.

$$\bar{A}_x = \frac{\mu_x + \lambda}{\mu_x + \lambda + \delta} = \frac{.04 + .02}{.04 + .02 + .04} = \frac{.06}{.10} = 0.6$$

$$\bar{A}_y = \frac{\mu_y + \lambda}{\mu_y + \lambda + \delta} = \frac{.06 + .02}{.06 + .02 + .04} = \frac{.08}{.12} = 0.667$$

$$\bar{A}_{xy} = \frac{\mu_x + \mu_y + \lambda}{\mu_x + \mu_y + \lambda + \delta} = \frac{.04 + .06 + .02}{.04 + .06 + .02 + .04} = \frac{.12}{.16} = 0.75$$

APV of prem = APV of benefits

$6.25P = \bar{A}_x + \bar{A}_y - \bar{A}_{xy} = 0.6 + 0.667 - 0.75$

$\boldsymbol{P = 0.082667}$

7. For a special fully continuous last-survivor insurance of 1 on two independent lives, (x) and (y), you are given:
   i) Death benefits are payable at the moment of the second death.
   ii) Level benefit premiums, π, are payable only while (x) is alive and (y) is dead. No premiums are payable while both are alive or if (x) dies first.
   iii) $\delta = 0.05$
   iv) $\mu_x(t) = 0.03, \ t \geq 0$
   v) $\mu_y(t) = 0.04, \ t \geq 0$

   Calculate π.

Solution:

Note that the premium annuity is a reversionary annuity to (x) after (y)'s death.
APV of premium $= \pi(\bar{a}_x - \bar{a}_{xy})$
Under constant force,

$$\bar{a}_x = \frac{1}{\mu_x + \delta} = \frac{1}{.03 + .05} = 12.5$$

$$\bar{a}_{xy} = \frac{1}{\mu_x + \mu_y + \delta} = \frac{1}{.03 + .04 + .05} = 8.33333$$

APV of benefits: $\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$

$$\bar{A}_x = \frac{\mu_x}{\mu_x + \delta} = \frac{.03}{.03 + .05} = \frac{3}{8} = 0.375$$

$$\bar{A}_y = \frac{\mu_y}{\mu_y + \delta} = \frac{.04}{.04 + .05} = \frac{4}{9} = 0.444$$

$$\bar{A}_{xy} = \frac{\mu_x + \mu_y}{\mu_x + \mu_y + \delta} = \frac{.03 + .04}{.03 + .04 + .05} = \frac{7}{12} = 0.58333$$

$\pi(\bar{a}_x - \bar{a}_{xy}) = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$

$\pi(12.5 - 8.33333) = 0.375 + 0.444 - 0.58333$

$\pi(4.166667) = 0.23611111$

$\pi = \mathbf{0.566}$

8. An annuity is payable at the beginning of each year, as defined below:
   - 1 per year while both (40) and (50) are alive
   - 2/3 per year after the first death (either (40) or (50))
   - 0 per year if both are dead

   You are given:
   i) Both lives are independent, having mortality that follows the Illustrative Life Tables with $i = 0.06$
   ii) $Var(\ddot{a}_{\overline{K(\overline{xy})|}}) = 8.253$

   Calculate the variance of this special annuity.

   Solution:

   Let Z = the annuity's present value random variable

   $$Z = 1 \cdot \ddot{a}_{\overline{K(xy)|}} + \frac{2}{3}(\ddot{a}_{\overline{K(\overline{xy})|}} - \ddot{a}_{\overline{K(xy)|}})$$
   $$= \frac{2}{3}\ddot{a}_{\overline{K(\overline{xy})|}} + \frac{1}{3}\ddot{a}_{\overline{K(xy)|}}$$

   Re: $Var(aS + bT) = a^2 Var(S) + b^2\, Var(T) + 2ab Cov(S, T)$

   $$Var(Z) = Var(\frac{2}{3}\ddot{a}_{\overline{K(\overline{xy})|}} + \frac{1}{3}\ddot{a}_{\overline{K(xy)|}})$$
   $$= \frac{4}{9}\, Var(\ddot{a}_{\overline{K(\overline{xy})|}}) + \frac{1}{9}\, Var(\ddot{a}_{\overline{K(xy)|}}) + \frac{4}{9} Cov(\ddot{a}_{\overline{K(\overline{xy})|}}, \ddot{a}_{\overline{K(xy)|}})$$

   $$Var(\ddot{a}_{\overline{K(xy)|}}) = \frac{{}^2A_{xy} - (A_{xy})^2}{d^2} = \frac{.12380 - (.29368)^2}{(.06/1.06)^2} = 11.72041442$$

   $$Cov(\ddot{a}_{\overline{K(\overline{xy})|}}, \ddot{a}_{\overline{K(xy)|}}) = Cov(\ddot{a}_{\overline{K(x)|}}, \ddot{a}_{\overline{K(y)|}}) + \frac{(A_x - A_{xy})(A_y - A_{xy})}{d^2}$$
   $$= 0 + \frac{(.16132 - .29368)(.24905 - .29368)}{(.06/1.06)^2} \qquad // \; Cov(\ddot{a}_{\overline{K(x)|}}, \ddot{a}_{\overline{K(y)|}}) = 0, \text{ under indep.}$$
   $$= 1.84371112$$

   Therefore,
   $$Var(Z) = \frac{4}{9}\, Var(\ddot{a}_{\overline{K(\overline{xy})|}}) + \frac{1}{9}\, Var(\ddot{a}_{\overline{K(xy)|}}) + \frac{4}{9} Cov(\ddot{a}_{\overline{K(\overline{xy})|}}, \ddot{a}_{\overline{K(xy)|}})$$
   $$= \frac{4}{9}(8.253) + \frac{1}{9}(11.72041442) + \frac{4}{9}(1.84371112)$$
   $$= \mathbf{5.789695434}$$

9. You are given:
   i) (x) is subject to a uniform distribution of deaths over each year of age
   ii) (y) is subject to a constant force of mortality of 0.25.
   iii) $q_{\overset{1}{x}y} = 0.125$
   iv) $T(x)$ and $T(y)$ are independent.
   v) $i = 0.05$

   Calculate $A^{\ 1}_{x:\overline{1|}}$.

   Solution:

   Re: $q_x = \int_0^1 {}_tp_x\, \mu(x+t)\, dt$ and ${}_tp_y = e^{-\mu t}$

   $$0.125 = q_{\overset{1}{x}y} = \int_0^1 {}_tp_y\, {}_tp_x\, \mu(x+t)\, dt = q_x \int_0^1 {}_tp_y\, dt = q_x \int_0^1 e^{-.25t}\, dt = q_x \left[\frac{1-e^{-.25}}{.25}\right]$$

   $\rightarrow q_x = 0.125 / 0.884796868 = 0.141275365$

   $A^{\ 1}_{x:\overline{1|}} = vq_x = (1.05)^{-1}(0.141275365) = \mathbf{0.134547966}$

10. For two lives (50) and (60) with independent future lifetimes:
    i) $\mu_{50}(t) = 0.002t$
    ii) $\mu_{60}(t) = 0.003t$

    Calculate ${}_{20}q_{\overset{1}{50}:60} - {}_{20}q_{50:\overset{2}{60}}$.

    Solution:

    $${}_{20}q_{\overset{1}{50}:60} - {}_{20}q_{50:\overset{2}{60}} = {}_{20}q_{50}\ {}_{20}p_{60}$$

    $${}_{20}q_{50} = 1 - \exp\left(-\int_0^{20} 0.002t\,dt\right)$$
    $$= 1 - e^{-0.001(20)^2} = 1 - 0.670320 = 0.329680$$
    $${}_{20}p_{60} = \exp\left(-\int_0^{20} 0.003t\,dt\right)$$
    $$= e^{-0.0015(20)^2} = 0.548812$$

    ${}_{20}q_{50}\ {}_{20}p_{60} = (0.329680)(0.548812) = \mathbf{0.180932}$